Phase transitions in systems possessing shock solutions

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Recently it has been shown that there are three families of stochastic one-dimensional nonequilibrium lattice models for which the single-shock measures form an invariant subspace of the states of these models. Here, both the stationary states and dynamics of single-shocks on a one-dimensional lattice are studied. This is done for both an infinite lattice and a finite lattice with boundaries. It is seen that these models possess both static and dynamical phase transitions. The static phase transition is the well-known low-high density phase transition for the asymmetric simple exclusion process. The branching-coalescing random walk and asymmetric Kawasaki-Glauber process models also show the same phase transition. Double-shocks on a one-dimensional lattice are also investigated. It is shown that at the stationary state the contribution of double-shocks with higher width becomes small, and the main contribution comes from thin double-shocks.

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I. INTRODUCTION

Reaction-diffusion systems is a well-studied area. People have studied reaction-diffusion systems using analytical techniques, approximation methods, and simulations. A large fraction of exact results belong to low-dimensional (specially one-dimensional) systems, where solving low-dimensional systems should in principle be easier. Despite their simplicity, these systems exhibit a rich and rather nontrivial dynamical and stationary behavior. Studies on the models far from equilibrium have shown that there is a remarkably rich variety of critical phenomena [1].

Shocks in one-dimensional reaction-diffusion models have received much interest recently [2–12]. There are some exact results about shocks in one-dimensional reaction-diffusion models together with simulations, numeric results [7], and also mean field results [3]. Formation of localized shocks in one-dimensional driven diffusive systems with spacially homogeneous creation and annihilation of particles has been studied in [13]. Recently, in [5], the families of models with traveling wave solutions on a finite lattice have been presented. These models are the asymmetric simple exclusion process (ASEP), the branching-coalescing random walk (BCRW), and the asymmetric Kawasaki-Glauber process (AKGP). In all of these cases the time evolution of the shock measure is equivalent to that of a random walker on a lattice with L sites with homogeneous hopping rates in the bulk and special reflection rates at the boundary.

Shocks have been studied at both the macroscopic and the microscopic levels and there are some efforts on addressing the question of how these macroscopic shocks originate from the microscopic dynamics [8]. Hydrodynamic limits are also investigated.

Among the important aspects of reaction-diffusion systems is the phase structure of the system. The static phase structure concerns the time-independent profiles of the system, while the dynamical phase structure concerns the evolution of the system, specially its relaxation behavior. In [14–19], the phase structure of some classes of single-or multiple-species reaction-diffusion systems have been investigated. These investigations were based on the one-point functions of the systems.

Here we study both stationary and also dynamics of the single-shocks on a one-dimensional lattice. This is done for both an infinite lattice and a finite lattice with boundaries. In the stationary state, the system can be found in the low-density or in the high-density phase. This phase transition is a well-known first order phase transition in familiar ASEP, and also its extensions [20]. The BCRW and AKGP models show the same phase transitions. We also investigate the dynamical phase transitions of the models. It is seen that ASEP has no dynamical phase transition, but both of the models BCRW and AKGP have three phases, and the system may show dynamical phase transitions. Double-shocks on a one-dimensional lattice have been also investigated, and its stationary behavior has been studied. It is shown that, in the thermodynamic limit, contribution of double-shocks with higher width become vanishingly small.

II. FIXING THE NOTATIONS

Consider a one-dimensional lattice, each point of which is either empty or contains one particle. Let the lattice have L sites. An empty state is denoted by $|0\rangle$ and an occupied state is denoted by $|1\rangle$.

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

If the probability that the site i is occupied is ρ_i then the state of that is represented by

$$\begin{pmatrix} 1-\rho_i \\ \rho_i \end{pmatrix}.$$

The observables of a reaction-diffusion system are the operators N_i^α, where i with $1 \leqslant i \leqslant L$ denotes the site number, and $\alpha = 0, 1$ denotes the hole or the particle: N_i^0 is the hole (vacancy) number operator at site i, and N_i^1 is the particle number operator at site i. One has obviously the constraint

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$$s_\alpha N_i^\alpha = 1, \tag{2}$$

where $\langle s|$ is a covector of the components of which (s_α's) are all equal to one. The constraint (2) simply says that each site is either occupied by one particle or empty. A representation for these observables is

$$N_i^\alpha := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes N^\alpha \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-i}, \tag{3}$$

where N^α is a diagonal 2×2 matrix the only nonzero element of which is the αth diagonal element, and the operators 1 in the above expression are also 2×2 matrices. The state of the system is characterized by a vector

$$|\mathbb{P}\rangle \in \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{L}, \tag{4}$$

where $\mathbb{V}$ is a two-dimensional vector space. All the elements of the vector $|\mathbb{P}\rangle$ are non-negative, and

$$\langle \mathbb{S}|\mathbb{P}\rangle = 1. \tag{5}$$

Here $\langle \mathbb{S}|$ is the tensor-product of L covectors $\langle s|$. The evolution of the state of the system is given by

$$|\dot{\mathbb{P}}\rangle = \mathcal{H}|\mathbb{P}\rangle, \tag{6}$$

where the Hamiltonian $\mathcal{H}$ is stochastic, by which it is meant that its nondiagonal elements are non-negative and

$$\langle \mathbb{S}|\mathcal{H} = 0. \tag{7}$$

Two conventions are used to write the master equation,

(a) $|\dot{\mathbb{P}}\rangle = -\mathcal{H}'|\mathbb{P}\rangle \Rightarrow |\mathbb{P}\rangle(t) = \exp(-t\mathcal{H}')|\mathbb{P}\rangle(0)$

(b) $|\dot{\mathbb{P}}\rangle = \mathcal{H}|\mathbb{P}\rangle \Rightarrow |\mathbb{P}\rangle(t) = \exp(t\mathcal{H})|\mathbb{P}\rangle(0)$.

These two conventions are related to each other simply through $\mathcal{H}' = -\mathcal{H}$. In the (a) convention nondiagonal elements of $\mathcal{H}'$ are negative of reaction rates, hence nonpositive, and its diagonal elements are non-negative. In the (b) convention nondiagonal elements of $\mathcal{H}$ are reaction rates, hence non-negative, and its diagonal elements are nonpositive. In the (a) convention the real parts of the eigenvalues of $\mathcal{H}'$ are non-negative, and the eigenvalue with minimum nonzero real part corresponds to the relaxation time. In the (b) convention the real parts of the eigenvalues of $\mathcal{H}$ are nonpositive, and the eigenvalue with the maximum nonzero real part corresponds to the relaxation time, and finally the state vector $|\mathbb{P}\rangle$ is the same in two conventions, and in both conventions the elements of $|\mathbb{P}\rangle$ are non-negative. Throughout this paper we use the (b) convention.

The interaction is nearest neighbor if the Hamiltonian is of the form

$$\mathcal{H} = \sum_{i=1}^{L-1} H_{i,i+1}, \tag{8}$$

where

$$H_{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes H \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-1-i}. \tag{9}$$

(It has been assumed that the sites of the system are identical, that is, the system is translation-invariant. Otherwise H in the right-hand side of Eq. (9) would depend on i.) The two-site Hamiltonian H is stochastic, that is, its nondiagonal elements are non-negative, and the sum of the elements of each of its columns vanishes:

$$(\langle s| \otimes \langle s|)H = 0. \tag{10}$$

Here H is a 4×4 matrix (as the system under consideration has two possible states in each site and the interactions are nearest neighbor). The nondiagonal elements of H are non-negative and equal to the interaction rates; that is, the element H_β^α with $\alpha \neq \beta$ is equal to the rate of change of the state β to the state α. α and β each represent the state of two adjacent sites. For example, if $\alpha = 01$ and $\beta = 10$, then H_β^α is the rate of particle diffusion to the right.

The evolution equation of one-point function $\langle n_i\rangle$ (k-point functions) depends on both one- and two-point functions ($k-1$-, k-, and $k+1$-point functions). Generally this set of evolution equations cannot be solved exactly. If one can obtain the state of the system $|\mathbb{P}\rangle$ exactly, there is no need to solve this set of evolution equations. In [21], a ten-parameter family of reaction-diffusion processes was introduced for which the evolution equation of k-point functions contains only k-or less-point functions. We call such systems autonomous. The average particle number in each site has been obtained exactly for these models. In [22,23], this has been generalized to multispecies systems and more-than-two-site interactions.

Although generally one cannot obtain the state of the system $|\mathbb{P}\rangle$ exactly, for a special choice of initial states and of course with some constraints on reaction rates, one may obtain the state of the system $|\mathbb{P}\rangle$. There are three families of stochastic one-dimensional nonequilibrium lattice models for which the single-shock measures are an invariant subspace of the states of these models [5]. If the initial state of these models is a linear superposition of shock measures then one can obtain the state of the system $|\mathbb{P}\rangle$ exactly. These models are the asymmetric simple exclusion process (ASEP), the branching-coalescing random walk (BCRW), and the asymmetric Kawasaki-Glauber process (AKGP).

Let us consider a one-dimensional lattice with L sites. The interaction is nearest neighbor if the Hamiltonian is of the form

$$\mathcal{H} = b_1 \otimes \mathbf{1}^{\otimes(L-1)} + \left[\sum_{i=1}^{L-1} H_{i,i+1}\right] + \mathbf{1}^{\otimes(L-1)} \otimes b_L, \tag{11}$$

where $H_{i,i+1}$ denotes interaction in the bulk and b_1 and b_L denote the interactions at the boundaries,

$$b_1 := \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \quad b_L := \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}. \tag{12}$$

α and γ (δ and β) are the rates of injection and extraction at the first site (at the Lth site). Each site may be occupied or vacant. We represent any configuration of the system by the vector $|E_a\rangle$. So the system is spanned by 2^L vectors, $|E_a\rangle$ ($a = 1, 2, \ldots, 2^L$), and any physical state is a linear combination of these vectors

$$|\mathbb{P}\rangle = \sum_{a=1}^{2^L} \mathcal{P}_a |E_a\rangle, \quad \text{where} \quad \sum_{a=1}^{2^L} \mathcal{P}_a = 1. \tag{13}$$

$\mathcal{P}_a$'s are non-negative real numbers. $\mathcal{P}_a$ is the probability of finding the system in the configuration (a).

It is said that the state of the system is a single-shock at the site k if there is a jump in the density at the site k and the state of the system is represented by a tensor product of the states at each site as

$$|e_k\rangle = u^{\otimes k} \otimes v^{\otimes (L-k)}, \tag{14}$$

where

$$u := \begin{pmatrix} 1-\rho_1 \\ \rho_1 \end{pmatrix}, \quad v := \begin{pmatrix} 1-\rho_2 \\ \rho_2 \end{pmatrix}. \tag{15}$$

It is seen that

$$\langle \mathbb{S}|e_k\rangle = 1. \tag{16}$$

$|e_k\rangle$ represents a state for which the occupation probability for the first k sites is ρ_1, and the occupation probability for the next $L-k$ sites is ρ_2. The set $|e_k\rangle, k=0,1,\ldots,L$ is not a complete set, but linearly independent.

There are three families of stochastic one-dimensional nonequilibrium lattice models, (ASEP, BCRW, AKGP), for which if the initial state of these models is a linear superposition of shock states, at the later times the state of the system $|\mathbb{P}\rangle$ remains a linear combination of shock state. For these models

$$\mathcal{H}|e_k\rangle = d_1|e_{k-1}\rangle + d_2|e_{k+1}\rangle - (d_1+d_2)e_k, \tag{17}$$

where d_i's are some parameters depending on the reaction rates in the bulk, ρ_1 and ρ_2. So the span of $|e_k\rangle$'s is an invariant subspace of $\mathcal{H}$, the Hamiltonian of the above-mentioned models. It should be noted that the number of $|e_k\rangle$'s are $L+1$, and any physical state is not necessarily expressible in terms of $|e_k\rangle$'s. For a finite lattice with the injection and extraction at the boundaries besides Eq. (17), there are two other relations.

$$\mathcal{H}|e_0\rangle = D_1|e_1\rangle - D_1|e_0\rangle,$$

$$\mathcal{H}|e_L\rangle = D_2|e_{L-1}\rangle - D_2|e_L\rangle, \tag{18}$$

where D_i's are two parameters generally depending on ρ_1 and ρ_2, and the reaction rates.

Let us assume that the initial state of the system is a linear combination of shock state

$$|\mathbb{P}\rangle(0) = \sum_{k=0}^{L} p_k(0)|e_k\rangle. \tag{19}$$

p_k's are not necessarily non-negative, and so any of them may be greater than one. For such an initial state, the system remains in the subspace spanned by shock measures.

$$|\mathbb{P}\rangle(t) = \sum_{k=0}^{L} p_k(t)|e_k\rangle. \tag{20}$$

Using Eq. (16), it is seen that

$$\sum_{k=0}^{L} p_k(t) = 1. \tag{21}$$

But it should be noted that these are not probabilities. p_k only expresses the contribution of a shock at the site k in the state of the system. Any shock state $|e_k\rangle$ can be expanded in terms of $|E_a\rangle$,

$$|e_k\rangle = \sum_{a=1}^{2^L} \Lambda_{ka}|E_a\rangle, \tag{22}$$

where the elements of $\mathbf{\Lambda}$ are non-negative. Substituting Eq. (22) into Eq. (20) and comparing with Eq. (13) gives

$$\mathcal{P}_a = \sum_{k=1}^{L} p_k \Lambda_{ka}. \tag{23}$$

Here $\mathcal{P}_a$ is the probability to find the system in the state $|E_a\rangle$, and so it is a non-negative number. The condition of non-negativeness of probabilities ($\mathcal{P}_a$'s) leads to constraints on p_k, see Eq. (23).

The three models are classified as follows.

(1) ASEP. The only nonvanishing rates in the bulk are the rates of diffusion to the right ω_{23} and diffusion to the left ω_{32}. For a finite lattice, there may be injection and extraction rates at the boundaries. α and γ are the injection and extraction rates at the left boundary, and δ and β the injection and extraction rates at the right boundary. In this case the densities can take any value between 0 and 1 ($\rho_1 \neq 0,1$ and $\rho_2 \neq 0,1$). d_1 and d_2 are

$$d_1 = \frac{\rho_1(1-\rho_1)}{\rho_2-\rho_1}(\omega_{23}-\omega_{32}),$$

$$d_2 = \frac{\rho_2(1-\rho_2)}{\rho_2-\rho_1}(\omega_{23}-\omega_{32}). \tag{24}$$

It should be noted that the densities ρ_1 and ρ_2 are also related through

$$\frac{\rho_2(1-\rho_1)}{\rho_1(1-\rho_2)} = \frac{\omega_{23}}{\omega_{32}}. \tag{25}$$

So

$$d_1 = \frac{\rho_1}{\rho_2}\omega_{23}, \quad d_2 = \frac{\rho_2}{\rho_1}\omega_{32}. \tag{26}$$

The rates of injection and extraction at the boundaries are also related to the densities ρ_1 and ρ_2.

$$\rho_1(1-\rho_1)(\omega_{23}-\omega_{32}) = \alpha(1-\rho_1) - \gamma\rho_1,$$

$$-\rho_2(1-\rho_2)(\omega_{23}-\omega_{32}) = \delta(1-\rho_2) - \beta\rho_2. \tag{27}$$

The parameters D_1 and D_2 are obtained to be

$$D_1 = d_2 + \frac{\alpha(1-\rho_2)-\gamma\rho_2}{\rho_1-\rho_2} = d_2 + \frac{\alpha}{\rho_1} - \omega_{23},$$

$$D_2 = d_1 - \frac{\delta(1-\rho_1)-\beta\rho_1}{\rho_1-\rho_2} = d_1 + \frac{\delta}{\rho_2} - \omega_{32}. \tag{28}$$

(2) BCRW. The nonvanishing rates are coalescence (ω_{34} and ω_{24}), Branching (ω_{42} and ω_{43}), and diffusion to the left and right (ω_{32} and ω_{23}). The injection rate at the right boundary δ should be zero. The density ρ_1 can take any value between 0 and 1, but ρ_2 should be zero. These parameters are related through

$$\frac{\omega_{23}}{\omega_{43}} = \frac{1-\rho_1}{\rho_1},$$

$$\frac{\omega_{23}}{\omega_{43}} = \frac{\omega_{24}+\omega_{34}}{\omega_{42}+g a_{43}}. \tag{29}$$

The rates of injection and extraction at the boundaries are also related to the densities ρ_1, and the reaction rates,

$$\rho_1\omega_{23} - \rho_1(1-\rho_1)\omega_{32} - \rho_1^2\omega_{34} = \alpha(1-\rho_1) - \gamma\rho_1. \tag{30}$$

The parameters d_1 and d_2 are

$$d_1 = (1-\rho_1)\omega_{32} + \rho_1\omega_{34},$$

$$d_2 = \frac{\omega_{43}}{\rho_1}. \tag{31}$$

The parameters D_1 and D_2 are obtained to be

$$D_1 = \frac{\alpha}{\rho_1},$$

$$D_2 = d_1 - d_2(1-\rho_1) + \beta. \tag{32}$$

(3) AKGP. The nonvanishing rates are Death (ω_{12} and ω_{13}) and Branching to the left and right (ω_{42} and ω_{43}), and also diffusion to the left. ρ_1 should be equal to one, and ρ_2 should be zero. The extraction rate at the left boundary γ and the injection rate at the right boundary δ should be zero. The hoping parameters are $d_1=\omega_{13}$, $d_2=\omega_{43}$, and finally $D_1=\alpha$, and $D_2=\beta$.

Interchanging ρ_1 and ρ_2 is nothing but exchanging left and right. Changing ρ_i to $1-\rho_i$ is particle hole exchange.

III. SINGLE-SHOCK

A. Shock on an infinite lattice

Here, we want to consider the evolution of shock measures on an infinite lattice. Here $|e_l\rangle$ stands for a state with a shock at the site l. If the initial state is a linear combination of shock measures, at later times the state of the system should be also expressible in terms of shock measures.

$$|\mathbb{P}\rangle(t) = \sum_{k=-\infty}^{\infty} p_k(t)|e_k\rangle. \tag{33}$$

Calculating p_k, one can obtain any correlation function of number operators.

$$\langle n_i\rangle = \langle \mathbb{S}|n_i|\mathbb{P}\rangle = \sum_{k=-\infty}^{\infty} p_k(t)\langle \mathbb{S}|n_i|e_k\rangle, \tag{34}$$

where

$$\langle \mathbb{S}|n_i|e_k\rangle = \begin{cases} \rho_1, & i \leqslant k \\ \rho_2, & i > k. \end{cases} \tag{35}$$

So

$$\langle n_i\rangle = \rho_2 + (\rho_1-\rho_2)\mathcal{B}_i(t) \tag{36}$$

where

$$\mathcal{B}_i(t) = \sum_{k=i}^{\infty} p_k(t). \tag{37}$$

All other correlation functions of number operators can be obtained in terms of $\mathcal{B}_i$'s.

$$\langle n_i\rangle = \rho_2 + (\rho_1-\rho_2)\mathcal{B}_i(t),$$

$$\langle n_i n_j\rangle = \rho_2^2 + (\rho_1-\rho_2)\{\rho_2\mathcal{B}_i(t) + \rho_1\mathcal{B}_j(t)\},$$

$$\langle n_i n_j n_k\rangle = \rho_2^3 + (\rho_1-\rho_2)\{\rho_2^2\mathcal{B}_i(t) + \rho_2\rho_1\mathcal{B}_j(t) + \rho_1^2\mathcal{B}_k(t)\}$$

$$\vdots \tag{38}$$

where $i<j<k<\cdots$.

The evolution equation for the system is

$$\mathcal{H}|\mathbb{P}\rangle = \frac{d}{dt}|\mathbb{P}\rangle. \tag{39}$$

Knowing the action of $\mathcal{H}$ on $|e_k\rangle$, one can obtain the evolution equation for $p_k(t)$.

$$\dot{p}_k = d_1 p_{k+1} + d_2 p_{k-1} - (d_1+d_2)p_k. \tag{40}$$

Here we have used the linear independence of $|e_k\rangle$'s. Let us define the generating function

$$G(z,t) := \sum_{l=-\infty}^{\infty} P_l(t) z^l. \tag{41}$$

Then the evolution equation for $G(z,t)$ is

$$\dot{G} = [d_1 z^{-1} + d_2 z - (d_1+d_2)]G, \tag{42}$$

the solution for which is

$$G(z,t) = e^{[d_1 z^{-1} + d_2 z - (d_1+d_2)]t} G(z,0). \tag{43}$$

$G(z,0)$ can be determined using Eq. (41) and contributions of the shock measures in the initial state. The coefficients for the Laurent expansion of the generating function are $p_k(t)$'s:

$$p_k(t) = e^{-(d_1+d_2)t} \sum_{m=-\infty}^{\infty} \left(\frac{d_2}{d_1}\right)^{(k-m)/2} I_{k-m}(2\sqrt{d_1 d_2}\,t) p_m(0). \tag{44}$$

The above result is first obtained in [4]. At large times,

$$p_k(t) \sim \left(\frac{d_2}{d_1}\right)^{k/2} \frac{e^{[-(d_1+d_2)+2\sqrt{d_1 d_2}]t}}{\sqrt{t}}. \tag{45}$$

It is seen from the above equation that if $d_2<d_1$, the contribution of the shocks at the rightmost sites tend rapidly to

their final value, and obviously for $d_2 > d_1$ the contribution of the shocks at the leftmost sites arrive earlier to their final values. This expression seems to be unbounded for $k \to \pm\infty$. For any fixed t, this is true. However, it simply means that in order for this term to represent the leading term for some k, t must be greater than some T, which does depend on k.

B. Shocks on a lattice with the boundary

Now let us first consider a one-dimensional lattice with L sites. There are injection and extraction at the boundaries. Because the boundary terms (17) change to

$$\mathcal{H}|e_k\rangle = d_1|e_{k-1}\rangle + d_2|e_{k+1}\rangle - (d_1+d_2)|e_k\rangle,$$

$$\mathcal{H}|e_0\rangle = D_1|e_1\rangle - D_1|e_0\rangle,$$

$$\mathcal{H}|e_L\rangle = D_2|e_{L-1}\rangle - D_2|e_L\rangle, \tag{46}$$

the evolution equation for p_k's in the bulk are the same as that of infinite lattice, but here one should take care of boundary terms.

$$\dot{p}_k = d_1 p_{k+1} + d_2 p_{k-1} - (d_1+d_2)p_k, \quad k \neq 0,1,L-1,L,$$

$$\dot{p}_{L-1} = D_2 p_L + d_2 p_{L-2} - (d_1+d_2)p_{L-1},$$

$$\dot{p}_1 = d_1 p_2 + D_1 p_0 - (d_1+d_2)p_1,$$

$$\dot{p}_L = d_2 p_{L-1} - D_2 p_L,$$

$$\dot{p}_0 = d_1 p_1 - D_1 p_0. \tag{47}$$

Let us first consider the stationary case. This set of equations can be solved easily,

$$p_k = \left(\frac{d_2}{d_1}\right)^{k-1}\left(\frac{d_1}{D_2}\right)^{\delta_{k,L}}\left(\frac{d_2}{D_1}\right)^{\delta_{k,0}} p_1. \tag{48}$$

For any finite L, all the p_k's can be obtained in terms of p_1, and p_1 is also obtained using the normalization condition (21). In the thermodynamic limit ($L \to \infty$), it is seen that for $d_2 > d_1$, finiteness of p_k for large k leads to vanishingly small p_1. But as d_1 exceeds d_2, p_k for large k becomes vanishingly small in the thermodynamic limit. This is the static phase transition previously mentioned. The static phase transition is controlled by the reaction rates in the bulk and is independent of the rates at the boundaries. It is a discontinuous change of the behavior of the derivative of the stationary value of p_k at the end points, with respect to the reaction rates. This phase transition is a well-known first order phase transition for the familiar ASEP, and also its extensions [20]. The BCRW and AKGP models show the same phase transitions. In the stationary state, the system can be found in the low-density or in the high-density phase, or in other words the contribution of the leftmost or rightmost shocks may be negligible.

Now, return to the dynamics of the system. To proceed let us first introduce a change of variable

$$q_k := p_k, \quad k \neq 0,L,$$

$$q_0 := (D_1/d_2)p_0,$$

$$q_L := (D_2/d_1)p_L. \tag{49}$$

Using this change of variable Eq. (47) recasts to

$$\dot{q}_k = d_1 q_{k+1} + d_2 q_{k-1} - (d_1+d_2)q_k, \quad k \neq 0,L$$

$$\dot{q}_L = \frac{d_2 D_2}{d_1} q_{L-1} - D_2 q_L,$$

$$\dot{q}_0 = \frac{d_1 D_1}{d_2} q_1 - D_1 q_0. \tag{50}$$

To find the relaxation of the system towards its stationary state, one should find the greatest nonzero eigenvalue of the operator h, defined through $\dot{q}_k =: h_k^l q_l$. The eigenvalues and eigenvectors of h have been denoted by E and $\mathbf{C}_E$, respectively. Expanding the vector q in terms of $\mathbf{C}_E$'s, and regarding the completeness and linear independency of $\mathbf{C}_E$'s, one arrives at

$$EC_k = d_1 C_{k+1} + d_2 C_{k-1} - (d_1+d_2)C_k, \quad k \neq 0,L,$$

$$EC_L = \frac{d_2 D_2}{d_1} C_{L-1} - D_2 C_L,$$

$$EC_0 = \frac{d_1 D_1}{d_2} C_1 - D_1 C_0. \tag{51}$$

The solution to the above equations is

$$C_k = a z_1^k + b z_2^k, \tag{52}$$

where z_i's satisfy

$$E = -(d_1+d_2) + d_1 z_i + \frac{d_2}{z_i}. \tag{53}$$

Then $z_1 z_2 = d_2/d_1$. The second and third equations of Eq. (51) take the form

$$(E+D_2)(a z_1^L + b z_2^L) - \frac{d_2 D_2}{d_1}(a z_1^{L-1} + b z_2^{L-1}) = 0,$$

$$(E+D_1)(a+b) - \frac{d_1 D_1}{d_2}(a z_1 + b z_2) = 0. \tag{54}$$

or

$$\left[(E+D_2)z_1 - \frac{d_2 D_2}{d_1}\right]a + \left[(E+D_2)z_2 - \frac{d_2 D_2}{d_1}\right]\left(\frac{z_2}{z_1}\right)^{L-1} b = 0,$$

$$\left[E + D_1 - \frac{d_1 D_1}{d_2} z_1\right]a + \left[E + D_1 - \frac{d_1 D_1}{d_2} z_2\right]b = 0. \tag{55}$$

To have nonzero solutions for q_k's, these equations should have nontrivial solutions for a and b, which means that the determinant of the coefficients should be zero,

$$\left[E + D_1 - \frac{d_1 D_1}{d_2} z_1\right]\left[(E + D_2) z_2 - \frac{d_2 D_2}{d_1}\right]\left(\frac{z_2}{z_1}\right)^{L-1} = \left[E + D_1 - \frac{d_1 D_1}{d_2} z_2\right]\left[(E + D_2) z_1 - \frac{d_2 D_2}{d_1}\right]. \quad (56)$$

Performing the change of variable $z_i =: \sqrt{d_2/d_1}\, x_i$ leads to

$$E = -(d_1 + d_2) + \sqrt{d_1 d_2}\,(x_i + x_i^{-1}) \quad (57)$$

and $x_1 x_2 = 1$. Equation (56) changes to

$$[E + D_1 - D_1 x_1 \sqrt{d_1/d_2}][(E + D_2) - D_2 x_1 \sqrt{d_2/d_1}]\, x_1^{-2L+2} = [(E + D_1) x_1 - D_1 \sqrt{d_1/d_2}][(E + D_2) x_1 - D_2 \sqrt{d_2/d_1}]. \quad (58)$$

Equation (58) can be written as a polynomial equation of order $2L$, so it has $2L$ solutions. Two obvious solutions of Eq. (58) are $x = \pm 1$. But, these generally do not correspond to eigenvalues and eigenvectors. In fact for these solutions, x_1 and x_2 are the same, so that Eq. (52) should be modified to $C_k = (a + bk)(-1)^k$ and it is not difficult to see that these do not fulfill the boundary conditions unless $a = b = 0$. It will be shown that two cases may occur, either both solutions are phases then $|x_1| = |x_2| = 1$, or both of them are real. Except for the solutions ± 1, one of the real solutions is greater than one, which we take to be x_1. For the phase solution, $x = \exp(i\theta)$

$$E = -(d_1 + d_2) + 2\sqrt{d_1 d_2}\cos\theta. \quad (59)$$

Among these sets of eigenvalues the biggest one is $E = -(\sqrt{d_1} - \sqrt{d_2})^2$. So if there is no other solution except for the phase solutions, the relaxation time is

$$\tau_0 = \frac{1}{(\sqrt{d_1} - \sqrt{d_2})^2}. \quad (60)$$

Now, let us search for the real solutions, if those exist. Of course, by real solutions it is meant real solutions besides the trivial solutions ± 1. If x is a solution to Eq. (58), x^{-1} is another solution to it. So it is sufficient to seek the solutions with $|x| > 1$. In the thermodynamic limit, and for $|x| > 1$, Eq. (58) is simplified to

$$[(E + D_2) x - D_2 \sqrt{d_2/d_1}][(E + D_1) x - D_1 \sqrt{d_1/d_2}] = 0. \quad (61)$$

If such a solution exists, then

$$E = -(d_1 + d_2) + \sqrt{d_1 d_2}\,(x + x^{-1}) \quad (62)$$

for any positive x, E obtained from Eq. (62) is greater than the eigenvalue obtained from Eq. (59), and so the system relaxes to its stationary state slower. If all of the solutions for Eq. (58) are phases we call the system in the fast phase, and if there is a real solution, the system is in the slow phase. Depending on parameters, there may be more than one real solution, and the system may be in the slow or slower phase. The solution of the first bracket in Eq. (61) is $x = \sqrt{d_2/d_1}$, and $(d_1 - D_2)/\sqrt{d_1 d_2}$. The first solution gives $E = 0$, which is related to the stationary state. We obtained these solutions assuming $|x| > 1$. So, $x = (d_1 - D_2)/\sqrt{d_1 d_2}$ is a solution provided that

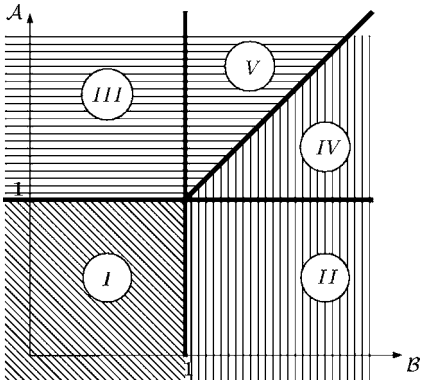

FIG. 1. -Exact phase diagram of the ASEP, BCRW, and AKGP, in the $\mathcal{A}$, $\mathcal{B}$ plane. For the ASEP, the parameter space is restricted to the region *I*. For the models BCRW and AKGP, the parameter space is restricted to the regions *I*, *II*, and *III*.

$$\mathcal{A} := \frac{d_1 - D_2}{\sqrt{d_1 d_2}} > 1. \quad (63)$$

Similarly there may be a solution for the second bracket in Eq. (61), $x = (d_2 - D_1)/\sqrt{d_1 d_2}$, provided that

$$\mathcal{B} := \frac{d_2 - D_1}{\sqrt{d_1 d_2}} > 1. \quad (64)$$

The above results are summarized in Fig. 1. Depending on the magnitudes of $\mathcal{A}$,$\mathcal{B}$, there are five regions. If both $\mathcal{A}$,$\mathcal{B}$ are less than one, then the relaxation time is given by Eq. (60) (region *I*). Changing continuously the reaction rates at the bulk or at the boundaries or the densities, the relaxation time may change discontinuously. This is the dynamical phase transition. If one of $\mathcal{A}$ and $\mathcal{B}$ become greater than one, then the relaxation time is given by $\tau_{\mathcal{A}}$ or $\tau_{\mathcal{B}}$

$$\tau_{\mathcal{A}} := \frac{d_1 - D_2}{D_2(d_1 - D_2 - d_2)},$$

$$\tau_{\mathcal{B}} := \frac{d_2 - D_1}{D_1(d_2 - D_1 - d_1)}. \quad (65)$$

If $\mathcal{A} < 1$ and $\mathcal{B} > 1$ the system is in the phase *II*. If $\mathcal{A} > 1$ and $\mathcal{B} < 1$ the system is in the phase *III*. And finally if it is possible that both $\mathcal{A}$ and $\mathcal{B}$ become greater than one, the relaxation time will be given by $\max(\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$. In the phase *IV*, the relaxation time is $\tau_{\mathcal{B}}$, and in the phase *V*, the relaxation time is $\tau_{\mathcal{A}}$.

Let us study each of the three models ASEP, BCRW, and AKGP separately.

(1) ASEP. To study the phase structure of this model, we should obtain $\mathcal{A}$ and $\mathcal{B}$. Let us assume $\omega_{32} < \omega_{23}$. Then $\omega_{32} < \sqrt{\omega_{23}\omega_{32}}$, and obviously

$$\omega_{32} - \frac{\delta}{\rho_2} < \sqrt{\omega_{23}\omega_{32}}.$$

Now using the second equation of Eq. (28), one arrives at

$$d_1 - D_2 < \sqrt{\omega_{23}\omega_{32}} = \sqrt{d_1 d_2} \Rightarrow \mathcal{A} < 1.$$

Now, let us assume $\omega_{23} < \omega_{32}$. Using Eq. (27) and the fact that β is the extraction rate and should be positive, one gets

$$\omega_{32} - \frac{\delta}{\rho_2} < \omega_{23}.$$

But $\omega_{23} < \sqrt{\omega_{23}\omega_{32}}$, so

$$d_1 - D_2 < \sqrt{\omega_{23}\omega_{32}} = \sqrt{d_1 d_2} \Rightarrow \mathcal{A} < 1.$$

With similar reasoning, one gets $\mathcal{B} < 1$. Therefore the only region in the space of parameters $\mathcal{A}$ and $\mathcal{B}$ available for the ASEP is the region *I*. So, there is no dynamical phase transition for ASEP, and the relaxation time is given by

$$\tau = \frac{\rho_1 \rho_2}{(\rho_1\sqrt{\omega_{23}} - \rho_2\sqrt{\omega_{32}})^2}. \tag{66}$$

(2) BCRW. Let us first define

$$\Omega := \omega_{32} + \frac{\rho_1}{1-\rho_1}\omega_{34}. \tag{67}$$

Using Eq. (30), and the fact that γ is the extraction rate, and should be positive, one gets

$$\frac{\alpha(1-\rho_1)}{\rho_1} > \omega_{23} - (1-\rho_1)\Omega. \tag{68}$$

Three cases may occur.

(i) $\omega_{23} < \Omega$ and $\alpha(1-\rho_1)/\rho_1 < \omega_{23} - (1-\rho_1)\sqrt{\omega_{23}\Omega}$. Then $\mathcal{A} < 1$ and $\mathcal{B} > 1$. Therefore the system is in the phase *II*, and the relaxation time is given by $\tau_{\mathcal{B}}$.

(ii) $\omega_{23} > \Omega$ and $\beta < \omega_{23} - (1-\rho_1)\Omega$. Then $\mathcal{A} > 1$ and $\mathcal{B} < 1$. Therefore the system is in the phase *III*, and the relaxation time is given by $\tau_{\mathcal{A}}$.

(iii) Otherwise the system is in the phase *I*, and the relaxation time is given by τ_0, Eq. (60).

The regions with $\mathcal{A} > 1$ and $\mathcal{B} > 1$ are not available for the BCRW. In summary, for the model BCRW, the parameter space is restricted to the regions *I*, *II*, and *III*. So, there are three distinct phases available for the system, and this model may experience dynamical phase transitions.

(3) AKGP. For this model we have

$$\mathcal{A} := \sqrt{\frac{\omega_{13}}{\omega_{43}}} - \frac{\beta}{\omega_{13}\omega_{43}},$$

$$\mathcal{B} := \sqrt{\frac{\omega_{43}}{\omega_{13}}} - \frac{\alpha}{\omega_{13}\omega_{43}}. \tag{69}$$

If $\omega_{13} > \omega_{43}$, then $\mathcal{B} < 1$. Depending on the rates ω_{13} and ω_{43}, and β, the system may be in phases *I* or *III*. And if $\omega_{13} < \omega_{43}$, then $\mathcal{A} < 1$. Depending on the rates ω_{13} and ω_{43}, and α, the system may be in phases *I* or *II*. For the model AKGP, the parameter space is restricted to the regions *I*, *II*, and *III*. So, there are three distinct phases available for the system, and the model AKGP may experience dynamical phase transitions.

IV. DOUBLE-SHOCK

We define the state of a double shock on a one-dimensional lattice with L sites as

$$|e_{m,k}\rangle = u^{\otimes m} \otimes v^{\otimes k} \otimes w^{\otimes(L-k-m)}, \quad m+k \leqslant L, \tag{70}$$

where

$$u := \begin{pmatrix} 1-\rho_1 \\ \rho_1 \end{pmatrix}, \quad v := \begin{pmatrix} 1-\rho_2 \\ \rho_2 \end{pmatrix}, \quad w := \begin{pmatrix} 1-\rho_3 \\ \rho_3 \end{pmatrix}. \tag{71}$$

$|e_{m,k}\rangle$ represents a state for which the occupation probability for the first m sites is ρ_1, the occupation probability for the next k sites is ρ_2, and the occupation probability for remaining sites is ρ_3. It should be noted that this state represents a double shock, one shock at the site m, and the other one at the site $m+k$. We call k the width of double-shock. We also assume that the three densities ρ_1, ρ_2, and ρ_3 are different. The evolution of the shocks in ASEP starting from a measure with several shocks and extra particles at the shock positions is studied in [4]. In that model, the measure is

$$|e_{m,k}\rangle = u^{\otimes m} \otimes Z \otimes v^{\otimes k} \otimes Z \otimes w^{\otimes(L-k-m)}, \quad m+k \leqslant L, \tag{72}$$

where

$$Z := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{73}$$

It is shown that with a special choice of densities, the time evolution equation of this model is similar to that of some random walkers on a lattice.

Although the states $|e_{m,k}\rangle$, $(m,k=0,1,\ldots)$ do not construct a complete basis, they are linearly independent provided that k the width of double-shock can be zero only when $m=0$ or $m=L$. This means that there should be at least one site with the occupation probability ρ_2 between the sites with the occupation probability ρ_1, and the sites with the occupation probability ρ_3. For the single-shock linear independency of $|e_k\rangle$'s are obvious, but here u, v, and w are not linearly independent, So it is not obvious that $|e_{m,k}\rangle$'s are linearly independent. To prove $|e_{m,k}\rangle$'s are linearly independent, one must show that if

$$\sum_{k=0}^{L}\sum_{m=0}^{L-k} a^{mk}|e_{m,k}\rangle = 0, \tag{74}$$

then all a^{mk}'s must be zero. Let us define

$$\tilde{u} := (\rho_1 \quad \rho_1 - 1), \quad \tilde{v} := (\rho_2 \quad \rho_2 - 1), \quad \tilde{w} := (\rho_3 \quad \rho_3 - 1). \tag{75}$$

Multiplying Eq. (74) from the left hand side by $\langle s|^{\otimes L}$, one arrives at

$$\sum_{k=0}^{L}{}'\sum_{m=0}^{L-k} a^{mk} = 0. \tag{76}$$

The prime on the first summation denotes that $k=0$, only when $m=0$, or $m=L$. Now, multiplying Eq. (74) from the left-hand side by $\langle s|^{\otimes L-1}\tilde{u}$ and $\tilde{w}\otimes\langle s|^{\otimes L-1}$ one arrives at

$$\sum_{k=0}^{L}\sum_{m=0}^{L-k} a^{mk} - a^{L0} = 0,$$

$$\sum_{k=0}^{L}\sum_{m=0}^{L-k} a^{mk} - a^{00} = 0. \tag{77}$$

Eq. (76) together with Eq. (77) gives $a^{L0}=a^{00}=0$. Now, define $f^{r,n}$ through

$$f^{rn} \coloneqq \tilde{v}^{\otimes r} \otimes s^{\otimes n} \otimes \tilde{v}^{\otimes(L-r-n)}, \tag{78}$$

where r and n are non-negative numbers and $n+r \leqslant L$. Multiplying Eq. (74) from the left-hand side by $f^{r,1}$ and using $f^{r,1}|e_{m,k}\rangle = \delta_m^r \delta_k^1$ (for $k \neq 0$), one arrives at $a^{r1}=0$. Similarly, it can be shown that all the coefficients a^{mk} in Eq. (74) should be zero. So, the set $|e_{m,k}\rangle$'s are linearly independent.

It was previously mentioned that there are three families of stochastic one-dimensional nonequilibrium lattice models (ASEP, BCRW, AKGP) for which if the initial state is a linear superposition of single-shock measures, at later times the state of the system $|\mathbb{P}\rangle$ remains a linear combination of shock measures. Among these, ASEP is the only model for which double-shocks form an invariant subspace, which means that if the initial state is a linear superposition of the double-shock measures, at later times the state of the system $|\mathbb{P}\rangle$ remains a linear combination of double-shocks. It can be shown that for the case of ASEP and on a one-dimensional lattice with infinite sites,

$$\mathcal{H}|e_{m,k}\rangle = d_1|e_{m-1,k+1}\rangle + d_2|e_{m+1,k-1}\rangle + d_3|e_{m,k-1}\rangle + d_4|e_{m,k+1}\rangle - (d_1+d_2+d_3+d_4)|e_{m,k}\rangle, \quad k \geqslant 2,$$

$$\mathcal{H}|e_{m,1}\rangle = d_1|e_{m-1,2}\rangle + d_4|e_{m,2}\rangle - (d_1+d_4)|e_{m,1}\rangle, \tag{79}$$

where

$$\frac{\omega_{23}}{\omega_{32}} = \frac{\rho_2(1-\rho_1)}{\rho_1(1-\rho_2)} = \frac{\rho_3(1-\rho_2)}{\rho_2(1-\rho_3)}, \tag{80}$$

and

$$d_1 = (\omega_{23}-\omega_{32})\frac{\rho_1(1-\rho_1)}{\rho_2-\rho_1},$$

$$d_2 = (\omega_{23}-\omega_{32})\frac{\rho_2(1-\rho_2)}{\rho_2-\rho_1},$$

$$d_3 = (\omega_{23}-\omega_{32})\frac{\rho_2(1-\rho_2)}{\rho_3-\rho_2},$$

$$d_4 = (\omega_{23}-\omega_{32})\frac{\rho_3(1-\rho_3)}{\rho_3-\rho_2}. \tag{81}$$

If the initial state is a linear combination of double-shocks, then

$$|\mathbb{P}\rangle(t) = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty} p_{mk}(t)|e_{mk}\rangle, \tag{82}$$

where $p_{m,k}$ is the contribution of the double-shock, and mk in the state of the system. Using Eqs. (6) and (82), and also the linear independency of $|e_{m,k}\rangle$'s, one can obtain the evolution equation for p_{mk}'s.

$$\dot{p}_{m,k} = d_1 p_{m+1,k-1} + d_2 p_{m-1,k+1} + d_3 p_{m,k+1} + d_4 p_{m,k-1} - (d_1+d_2+d_3+d_4)p_{m,k}, \quad k \geqslant 2,$$

$$\dot{p}_{m,1} = d_2 p_{m-1,2} + d_3 p_{m,2} - (d_1+d_4)p_{m,1}. \tag{83}$$

Let us define

$$q_k \coloneqq \sum_{m=-\infty}^{\infty} p_{mk}. \tag{84}$$

Then q_k is the contribution of all double-shocks with the width, the distance between two shocks, k in the state of the system. Conservation of the probability, Eq. (5), leads to

$$\sum_{k=1}^{\infty} q_k = 1. \tag{85}$$

The evolution equations for q_k's are

$$\dot{q}_k = D_1 q_{k-1} + D_2 q_{k+1} - (D_1+D_2)q_k, \quad k \geqslant 2,$$

$$\dot{q}_1 = D_2 q_2 - D_1 q_1, \tag{86}$$

where

$$D_1 \coloneqq d_1 + d_4, \quad D_2 \coloneqq d_2 + d_3. \tag{87}$$

D_1 and D_2 can be written in terms of ρ_1 and ρ_2 and the diffusion rates.

$$D_1 = (\omega_{32}-\omega_{23})\left\{\frac{\rho_1(1-\rho_1)}{\rho_1-\rho_2} + \frac{\rho_3(1-\rho_3)}{\rho_2-\rho_3}\right\},$$

$$D_2 = (\omega_{32}-\omega_{23})\left\{\frac{\rho_2(1-\rho_2)}{\rho_1-\rho_2} + \frac{\rho_2(1-\rho_2)}{\rho_2-\rho_3}\right\}. \tag{88}$$

Subtracting these

$$D_2 - D_1 = \frac{\rho_2(1-\rho_2)(1+F)(1-F)^2}{\omega_{23}[(1-\rho_2)+F\rho_2][F(1-\rho_2)+\rho_2]}, \tag{89}$$

where $F \coloneqq \omega_{32}/\omega_{23}$, it is seen that $D_1 < D_2$. One can easily obtain the steady state solution for q_k's.

$$q_k = \left(\frac{D_1}{D_2}\right)^{k-1} q_1,$$

$$q_1 = 1 - \frac{D_1}{D_2}. \tag{90}$$

To obtain the second equation of Eq. (90), we have used Eq. (85). Using the fact that $D_1 < D_2$, it is seen that q_k goes to zero for large k's. This means that at the stationary state the contributions of the double-shocks with larger width are less

and the main contribution comes from thin double-shocks. This is reminiscent of double-shocks in Burgers equation. In fact, in the hydrodynamic limit double-shock is not stable and converges to single-shock. Here, we show that in the microscopic level, and in the thermodynamic limit, the stationary value of the contribution of double-shocks with larger width becomes vanishingly small.

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